EULER CLASS AND FREE GENERATION

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"What can you say in your favour?"
"You see ..."
"Enough. Shoot'm. Next." [S].

This paper consists of two parts. In the first auxilliary part, we deal with sets with cyclic order. For any such set $\mathcal{O}$ we introduce following [BG] a cocycle $\ell : G \times G \to \mathbb{Z}$ valued in $\{0, \pm 1\} \subset \mathbb{Z}$ on the group G of automorphisms of $\mathcal{O}$. The cohomology class of ℓ in $H^2(G, \mathbb{Z})$ will be called the Euler class. If K is an ordered field, then the projective line $\mathbb{P}^1(K)$ has a cyclic order and $PSL_2(K)$ acts order-preserving on $\mathbb{P}^1(K)$, so that we get both the cocycle ℓ and the Euler class in $H^2(PSL_2(K), \mathbb{Z})$. If $K = \mathbb{R}$, then our Euler class coincides with the usual Euler class on $PSL_2(\mathbb{R})$.

In view of our extension of the Euler class to all ordered fields, the following two problems arise.

Problem 1. Let $\rho : \pi_1(S) \to PSL_2(K)$ be a homomorphism of the fundamental group of a closed oriented surface. Is it true, that $|(\rho^*[\ell], [S])| \leq 2g - 2$?

Problem 2. Suppose $|(\rho^*[\ell], [S])| = 2g - 2$. Is it true that ρ is injective?

For $\mathbb{K} = \mathbb{R}$ the theorems of Milnor [M] and Goldman [Go2], answer these problems positively.

In the main second part of this paper, we apply the cocycle ℓ and the ideas from the theory of Hamiltonian systems on the Teichmüller space to the following classical problem:

Problem 3. When n matrices in $SL_2(\mathbb{R})$ generate a free discrete group?

For $n = 2$ this problem has been treated in many papers, see [Gi]. An effective solution is given in [Gi]. The analogous problem for $SL_2(\mathbb{C})$ also attracted a lot of attention especially since Jörgensen paper [J]. For $n > 2$ however, the problem becomes much harder. We will give a simple *sufficient* condition for n hyperbolic elements in $SL_2(\mathbb{R})$ to generate a free discrete group. This condition is open, that is, satisfied on an open domain in $(SL_2(\mathbb{R}))^n$. Here is our main result.

Let n be even and let $a_i, b_i, 1 \leq i \leq n$ be in $SL_2(\mathbb{R})$. Suppose $h = \prod_{i=1}^n [a_i, b_i]$ is hyperbolic. Consider the eigenvectors x_1, x_2 of h and a matrix r which takes the

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form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the basis (x_1, x_2). Put $a_i = rb_{n+1-i}r^{-1}, b_i = ra_{n+1-i}r^{-1}$ for $n+1 \le i \le 2n$. Let $I_j = \prod_{i=1}^{j}[a_i, b_i] (j \le 2n)$.

Main Theorem. *Let* $f(a,b) = \frac{1}{\pi}\sum_{j=1}^{2n} \ell(I_{j-1}, a_j) + \ell(I_{j-1}a_j, b_j) - \ell(I_{j-1}a_j b_j a_j^{-1}, a_j) - \ell(I_j, b_j)$ *Then*

(a) $f(a,b)$ *is an integer and* $|f(a,b)| \le 2n-1$

(b) *if* $|f(a,b)| = 2n-1$, *then* $\{a_i, b_i\}$ *generate a free hyperbolic group in* $SL_2(\mathbb{R})$.

1. Cyclically ordered sets, ordered fields and the Euler class.

1.1. A cyclically ordered set $\mathcal{O}$ is a set with a subset Ω in $\mathcal{O} \times \mathcal{O} \times \mathcal{O}$, satisfying the following conditions:

(i) if $(x, y, z) \in \mathcal{O}$ then x, y, z are all different

(ii) if σ is a permutation of (x, y, z) and $(x, y, z) \in \mathcal{O}$ then $\{\sigma(x), \sigma(y), \sigma(z)\} \in \mathcal{O}$ if and only if σ is even

(iii) if z is fixed then the relation $x < y \Leftrightarrow (x, y, z) \in \mathcal{O}$ is a linear order.

1.2 Example. Let K be an ordered field and let $\mathbb{P}^1(K)$ be a projective line over K. We can think of $\mathbb{P}^1(K)$ as $K \cup \{\infty\}$. The cyclic order in $\mathbb{P}^(K)$ is defined by a condition that the induced order in K is standard. The group $PSL_2(K)$ acts on $\mathbb{P}^1(K)$ preserving the cyclic order.

1.3. Define a function $\psi : \mathcal{O} \times \mathcal{O} \times \mathcal{O} \to \{0, \pm 1\}$ in a following way:

(i) if any of (x, y, z) are equal, then $\psi(x, y, z) = 0$

(ii) ψ is odd under permutation of (x, y, z)

(iii) if $(x, y, z) \in \Omega$, then $\psi(x, y, z) = 1$.

1.4. Now let G be a group, acting in order preserving way on $\mathcal{O}$. Fix any element $p \in \mathcal{O}$ and define a function $\ell : G \times G \to \{0, \pm 1\}$ as $\ell(g_1, g_2) = \psi(p, g_2 p, g_1 g_2 p)$.

Lemma (1.4). *ℓ is an integer cocycle on G.*

Proof. is a direct computation and left to the reader.

Definition. The cohomology class of ℓ in $H^2(G)$ (which does not depend on p) is called the Euler class. In particular, for an ordered field K one gets the Euler class in $H^2(PSL_2(K), \mathbb{Z})$.

1.5 Comparison theorem ([BG]). *For $K = \mathbb{R}$, the class of ℓ in $H^2(PSL_2(\mathbb{R}))$ is the usual Euler class of associated S^1-bundle over $BPSL_2^\delta(\mathbb{R})$.*

Proof. Consider the action of $PSL_2(\mathbb{R})$ on $\mathcal{H}^2$. For any $p \in \mathcal{H}^2$ the class $\ell_p(g_1, g_2) =$ Area $(p, g_2 p, g_1 g_2 p)$ represents the Euler class e [Gu]. Here Area (p, q, r) is the area of oriented geodesic triangle with vertices in p, q, r. Now $[\ell_p] \in H^2(PSL_2(\mathbb{R}))$ does not depend on p and all cocycles ℓ_p are uniformly bounded. For $p_0 \in \partial\mathcal{H}^2$ and $p \to p_0$, we will have ℓ^∞-convergence of cocycles $\ell_{p_i} \to \ell_{p_0}$. Moreover the area of ideal triangle (p, q, r) is $\pi \cdot \psi(p, q, r)$. Any homology class in $H_2(PSL_2(\mathbb{R}))$ is represented by a map of a surface group $\pi_1(S) \xrightarrow{\alpha} PSL_2(\mathbb{R})$. It follows that $(\ell_{p_0}, \alpha_*[S]) = \lim(\ell_{p_i}, \alpha_*[S]) = (e, \alpha_*[S])$. This completes the proof.

2. Discrete Goldman twist. We will work with the representation variety $\mathcal{M} =$ Hom $(\pi_1(S), SL_2(\mathbb{R}))/SL_2(\mathbb{R})$, where S is an oriented closed surface of genus g. It

has $2^{2g+1}+2g-3$ connected components, indexed by the value of the Euler class [H]. Every such component, say $\mathcal{M}_e$, is a symplectic manifold, nonsingular if $e \neq 0$. For any γ a conjugacy class in $\pi_1(M)$, there is a natural Hamiltonian $Tr_\gamma : \mathcal{M} \to \mathbb{R}$, and the corresponding Hamiltonian flow has been identified by Goldman. If γ can be represented by a simple separating curve, this flow can be described as follows. Write a presentation of $\pi_1(S)$ in the following form:

$$[x_1, x_2] \dots [x_{2\kappa-1}, x_{2\kappa}] = [\gamma] = [x_{2\kappa+1}, x_{2\kappa+2}] \dots [x_{2g-1}, x_{2g}]$$

Next, write $[\gamma] = \exp A$ for some $A \in sl_2(\mathbb{R})$. Then put

$$\begin{aligned} \bar{x}_i^t &= \bar{x}_i, i \leq 2\kappa \\ \bar{x}_i^t &= \exp(-tA)\bar{x}_i \exp tA, i \geq 2\kappa; \end{aligned}$$

this is the flow of Tr_γ. In particular, $f_t : \{\bar{x}_i\} \to \{\bar{x}_i^t\}$ is a symplectomorphism of $\mathcal{M}$. Here $\bar{x}_i$ stands for the representation matrix of x_i.

For different γ, the Hamiltonians Tr_γ yield nice commutation relations, discovered by Wolpert [W] and put in a more "representation variety language" by Goldman [Go1]. In fact, the free module on the set of conjugacy classes of $\pi_1(M)$ becomes a Lie ring with Goldman's bracket. One may wonder what kind of group object correspond to it.

Whatever this eventual "Kac-Moody-Goldman" group may be, we will introduce now some elements from the "other connected components" of it. These are defined for $\mathcal{M}_{\pm(g-1)}$, which is naturally symplectomorphic to the Teichmüller space by a theorem of Goldman (see various proofs in [Go2], [H], [Re2]. In this case, all representation matrices are hyperbolic.

For γ as above, let r be a unique matrix (up to sign), commuting with $[\bar{\gamma}]$ with eigenvalues $+1$ and -1. Put

$$\begin{aligned} f(\bar{x}_i) &= x_i, i \leq 2\kappa \\ f(\bar{x}_i) &= r^{-1}\bar{x}_i r, i \geq 2\kappa \end{aligned}$$

This is a symplectic diffeomorphism of $\mathcal{M}_{\pm(g-1)}$. There is a particularly nice description of the map f if one views $\mathcal{M}_{g-1}$ as Teichmüller space. Namely, realise a point in $\mathcal{M}_{g-1}$ as a hyperbolic metric on S and find a geodesic, representing γ. We assume that the marked point lies on γ. Cut S into two pieces along γ and glue again by a reflection, which fixes a marked point. Then the new hyperbolic structure is the image of f.

3. Euler class. For a representation $x_i \to \bar{x}_i$ of $\pi_1(S)$ in $SL_2(\mathbb{R})$ the Euler number is an integer between $-(g-1)$ and $(g-1)$ by Milnor [M]. As mentioned above, all representations with the maximal Euler number are discrete faithful hyperbolic by Goldman's theorem. One can introduce a universal Euler class e in $H^2(SL_2^\delta(\mathbb{R}), \mathbb{R})$ as the image of a generator in continuous cohomology $H^2_{cont}(SL_2(\mathbb{R}), \mathbb{R})$. A representation $x_i \to \bar{x}_i$ as above defines a homology class in $H_2(SL_2^\delta(\mathbb{R}), \mathbb{Z})$, the image of the generator of $H_2(\pi_1(S), \mathbb{Z}) \approx \mathbb{Z}$, and the Euler number is just given by the evaluation of e on this class. Now, the generator of $H_2(\pi_1(S), \mathbb{Z})$ can be realized by an explicit cycle in the standard complex [B], that is, $\sum_{j=1}^{2g}(I_{j-1}|x_j) + (I_{j-1}x_j|y_j) - (I_{j-1}x_j y_j x_j^{-1}|y_j) - (I_j|y_j)$ where

$I_j = [x_1, y_1] \dots [x_j, y_j]$. On the other hand, the universal Euler class may be realized by a cocycle $A, B \mapsto \ell(A, B)$ as defined in Section 1. So the Euler number will be

$$\sum_{j=1}^{y} \ell(\bar{I}_{j-1}, \bar{x}_j) + \ell(\bar{I}_{j-1}, \bar{x}_j, \bar{y}_i) - \ell(\bar{I}_{j-1}\bar{x}_j\bar{y}_j\bar{x}_j^{-1}, \bar{y}_j) - \ell(\bar{I}_j, \bar{y}_j),$$

where $\bar{I}_j$ is defined in an obvious way.

3. Proof of the Main Theorem. Consider a closed surface S of genus $2n$. A map $x_i \to a_i, y_i \to b_i, 1 \le i \le 2n$ defines a homomorphism from $\pi_1(S)$ to $SL_2(\mathbb{R})$. Indeed, $\prod_{i=1}^{n}[a_i, b_i] \cdot \prod_{i=n+1}^{2n}[a_i, b_i] = h \cdot r^{-1}h^{-1}r = 1$. Next, the Euler number of this representation is computed as above, so $f(a, b)$ is always an integer. Moreover, if $f(a, b) = 4n - 2$, then the representation above is discrete and faithful. Q.E.D.

$SU(1, n)$-case, I. Consider a standard action of $SU(1, n)$ on the unit ball $B \subset \mathbb{C}^n$ with the complex hyperbolic metric. Let ω be the Kähler form of B. Fix a point ∞ in the sphere at infinity and consider a function $\varphi(A, B) = \psi(\infty, A(\infty), AB(\infty))$, where $\psi(x, y, z)$ is an integral of ω over any surface, spanning the geodesic triangle with vertices x, y, z. Let a_i, b_i, $1 \le i \le g$ be matrices in $SU(1, n)$. Let $h = \prod_{i=1}^{g}[a_i, b_i]$ and suppose h has a nonisotropic eigenvector. Then there exists a reflection r, commuting with h. Define a_i, b_i, $i \ge g + 1$ as in Introduction.

3.1 Theorem. *Define $f(a, b)$ by the formula in the Main Theorem. Then*

(a) *$f(a, b)$ is an integer and $|f(a, b)| \le 2g - 1$*
(b) *if $f(a, b) = 2g - 1$, then $\{a_i, b_i\}$ generate a discrete group in $SU(1, n)$.*

Proof. Same as above with Toledo's theorem [To], [Re1] instead of Goldman's.

4. Bounded cohomology. The bounded cohomology theory is an invention of Mikhail Gromov. The idea is as follows: in the standard complex, computing the real group cohomology we look only at bounded cochains, that is, bounded functions $f : \underbrace{G \times G \times \ldots \times G}_{i} \to \mathbb{R}$. The resulted cohomology spaces are called $H_b^i(G, \mathbb{R})$. There is a canonical homomorphism $H_b^i(G, \mathbb{R}) \to H^i(G, \mathbb{R})$.

4.1 Example. Let M be a symmetric space of negative curvature with isometry group G. Let ω be any G-invariant i-form on M. Then one gets a (Borel) class $Bor\,(\omega) \in H^i_{cont}(G, \mathbb{R})$ (see [Re 1] for example), which may be represented by bounded cocycle [Gr 1]. The Euler class in $SU(1, n)$, $n \ge 1$ is a further specialization.

4.2 Second bounded cohomology and combinatorial group theory. Consider a kernel of the map $H_b^2(G, \mathbb{R}) \to H^2(G, \mathbb{R})$. It gives rise to a function $f : G \to \mathbb{R}$ satisfying $|f(xy) - f(x) - f(y)| \le C$. Moreover, this function may be chosen a class function, that is, $f(xyx^{-1}) = f(y)$ and such that $f(x^n) = nf(x)$ [BG].

Next, for an element $z \in G' = [G, G]$ a *genus norm* is the smallest integer g such that z is a product of g generators. A theorem of Culler [C] states:

Theorem 4.2 (Culler). *If G is a f.g. free group. There is a positive constant such that for any $z \in G'$, $\| z^n \|_{genus} \ge const \cdot n$.*

The following result has been proved by Gromov [Gr2] and author [Re3].

Theorem 4.3. *Let G be geometrically hyperbolic, that is a fundamental group of a manifold of pinched negative curvature with $i(x) \underset{x\to\infty}{\longrightarrow} \infty$ (e.g. compact). Then the conclusion of the Theorem 4.2 holds.Here $i(x)$ is the injectivity radius at the point x.*

Now, we have

Proposition 4.4. *Let $f : G \to \mathbb{R}$ be as above. If $f(z) \neq 0$, then $\| z^n \|_{genus} \geq const \cdot n$.*

Proof. Let $z^n = \prod_{i=1}^{g}[x_i, y_i]$. Then

$$|n \cdot f(z)| = |f(z^n)| = |f(\prod_{i=1}^{g}[x_i, y_i])| = |\sum_{i=1}^{g} f([x_i, y_i])| +$$
$$+C \cdot g \leq 3c \cdot g + C \cdot g = 4C \cdot g, \quad \text{so} \quad g \geq \frac{n|f(z)|}{4C},$$

Q.E.D.

Genus norm in lattices in $SU(1, n)$.

Theorem (4.5). *Let $\Gamma \subset SU(1, n)$ be a lattice with $H_2(\Gamma, \mathbb{R}) = 0$. There exists a nonzero function $f : \Gamma \to \mathbb{R}$ as above such that if $f(z) \neq 0$, then $\| z^n \|_{genus} \geq const \cdot n$.*

Remarks. 1. Observe that if $f(z) \neq 0$, then for any $y \in G$ and κ big enough, $f(z^\kappa y) \neq 0$, so there are "many" z for which the Theorem 6.1 applies.
2. If Γ is cocompact or if $|Tr\,\gamma| \underset{\gamma\to\infty}{\longrightarrow} \infty$ in Γ then the conclusion of the Theorem follows from 4.3.

The proof of theorem 4.5 will be completed in 5.2.

5. Ergodic cocycle and measurable transfer. Let G be a locally compact group, H a closed subgroup and $X = G/H$. Suppose G has an invariant finite Borel measure μ on X. Suppose we have a measurable section $\delta : X \to G$. For any $g \in G$ and $x \in X$ we define $\lambda(g, x) \in H$ as unique element such that

$$s(gx) = gs(x)\lambda(g, x)$$

This defines a map of groups

$$G \xrightarrow{\lambda} H^X$$

Now, G acts on H^X by changing the argument. The map λ is well-known to be a *cocycle* for first non-abelian cohomology. Suppose $f(h_1, \ldots, h_n)$ is a measurable n-cocycle on H. Then the composition $f \circ \lambda : G^{(n)} \to \mathbb{R}$ is a measurable cocycle valued in the G-module of measurable functions on X. If f is bounded, so is $f \circ \lambda$. In this case, $\int_X f \cdot \lambda$ is a real bounded n-cocycle on G and we have a well-defined map

$$H_b^n(H, \mathbb{R}) \xrightarrow{t} H_b^n(G, \mathbb{R})$$

Moreover, the composition $H_b^n(G,\mathbb{R}) \xrightarrow{res} H_b^n(H,\mathbb{R}) \xrightarrow{t} H_b^n(G,\mathbb{R})$ is a multiplication by $\operatorname{Vol} X$.

The proof of all this facts is easily adopted from [Gr1].See [Re4].

5.2. As an immediate corollary, we state:

Theorem (5.2). *Let Γ be a lattice in either 1) $SO(n,1)$ or 2) $SU(n,1)$ or 3) $SU_{\mathbb{H}}(n,1)$. Then in case 1) $H_b^n(\Gamma) \neq 0$; in case 2) $H_b^{2\kappa}(\Gamma,\mathbb{R}) \neq 0$ for all $1 \leq \kappa \leq n$; in case 3) $H_b^{4\kappa}(\Gamma,\mathbb{R})$ for all $1 \leq \kappa \leq n$.*

Proof. Let us prove 2), since the rest is similar. $SU(n,1)$ acts isometrically on the complex ball B^n. The Kähler form ω defined, as in 4.1, a class ω in $H^2_{b,cont}(SU(n,1),\mathbb{R})$. It is nontrivial since for any cocompact Γ the restriction on $H^2(\Gamma)$ gives the Kähler class of B/Γ. Now for any Γ, the restriction on $H_b^2(\Gamma,\mathbb{R})$ must be nontrivial, otherwise $\mathrm{Vol}(SU(n,1)/\Gamma)\cdot\omega = 0$ even as a class in $H^2(SU^\delta(n,1),\mathbb{R})$.

Proof of the theorem 4.5. We need only to handle the case $H_2(\Gamma,\mathbb{R}) = 0$. Then the restriction of the just defined class in $H_b^2(\Gamma,\mathbb{R})$ on $H^2(\Gamma,\mathbb{R})$ is zero, so by 4.2 we have a function f with desired properties.

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